Renormalization Group analyses in $d = 2+\varepsilon$.

Above considerations suggest that in $d = 2+\varepsilon$ one obtains a stable SSB phase at low-T and therefore, an order-disorder transition as the temperature is increased (high-T phase has exponentially decaying correlations on general grounds).

The original action is

$$S = \frac{1}{2T}\int (\partial_\mu \vec{n})^2 \, d^d x \quad \text{where } d = 2+\varepsilon.$$

We decompose $\vec{n}$ as

slow fast

$$\vec{n} = \left(\cos\theta \sqrt{1-\vec{\sigma}^2} \, , \, \sin\theta \sqrt{1-\vec{\sigma}^2} \, , \, \vec{\sigma} \right)$$

where $\vec{\sigma}$ is $N-2$ component vector. This decomposition may seem a bit artificial – however, it is motivated by the fact that in $d=2$ (around which we are doing the expansion), $N=2$ component vector has slow fluctuations. Further, a completely different parametrization (briefly discussed below) yields the same result. Substituting this parametrization into the action, one finds,

$$S = \frac{1}{2T} \int d^d x \left\{ (1 - \vec{\sigma}^2)(\partial_\mu \theta)^2 + \left[\partial_\mu \left(\sqrt{1-\vec{\sigma}^2}\right)\right]^2 + (\partial_\mu \vec{\sigma})^2 \right\}$$

$$= \frac{1}{2T} \int d^d x \left\{ (1 - \vec{\sigma}^2)(\partial_\mu \theta)^2 + \frac{(\vec{\sigma} \cdot \partial_\mu \vec{\sigma})^2}{(1-\vec{\sigma}^2)} + (\partial_\mu \vec{\sigma})^2 \right\}$$

To perform Wilson RG, we will simply integrate out the $\vec{\sigma}$ field in a momentum shell to obtain renormalized coupling for θ.

Writing $\theta = \theta_> + \theta_<$, $\vec{\sigma} = \vec{\sigma}_> + \vec{\sigma}_<$

$$S \simeq \frac{1}{2T} \int d^d x \Big[(\partial_\mu \theta_<)^2 + (\partial_\mu \theta_>)^2 - (\vec{\sigma}_>^2 + \vec{\sigma}_<^2)\left((\partial_\mu \theta_<)^2 + (\partial_\mu \theta_>)^2 \right) + (\partial_\mu \sigma_<)^2 + (\partial_\mu \sigma_>)^2 \Big]$$

where we have dropped higher-derivative terms for $\vec{\sigma}$ such as $(\vec{\sigma}^2 \cdot \partial_\mu \vec{\sigma})^2$ since they are irrelevant in $2+\varepsilon$ dimension.

To find the RG flow of T, we simply integrate out $\sigma_>$ in $\sigma_>^2 (\partial_\mu \theta_<)^2$ to obtain the following action for $\theta_<$

$$\int_0^{\Lambda/b} \frac{k^2}{2T} |\theta_<(k)|^2 \, d^d k - \int_0^{\Lambda/b} \frac{k^2 |\theta_<(k)|^2}{2T} \langle \sigma_>^2(x) \rangle$$

$$- \frac{1}{2T} \int (\partial_\mu \theta_<)^2 \, \sigma_<^2(x) \, d^d x$$

$$\langle \vec{\sigma}_>^2 (x) \rangle = (n-2) T \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{k^2} \approx (n-2) K_d \Lambda^{d-2} \, dl$$

$$(b = e^{dl}).$$

Next, we need to rescale k and $\theta_<$ so that $k \in [0, \Lambda]$ and the action for $\theta_<$ is invariant in $d=2$. To find the rescaling factor for θ, its easier to work in real space.

$\frac{1}{T a^{d-2}} \int d^d x (\partial_\mu \theta)^2$ where we have added a factor of a^{d-2} to keep the action dimensionless.

Under $a \to ba$, $T \to b^{2-d}$ to keep the action invariant $\Rightarrow$ RG eigenvalue of T is $2-d$ consistent with the fact that the $T=0$ ordered phase is likely stable in $d>2$.

Rescaling $k \to bk$, and using the fact that RG eigenvalue of T is $2-d$ at $T=0$

$$\Rightarrow \quad \frac{1}{T(b)} = \frac{1}{T\, b^{2-d}} \left[1 - T(n-2) K_d \Lambda^{d-2}\, dl \right]$$

$$\frac{1}{T(1+dl)} = \frac{1+(d-2)\,dl}{T(1)} - (n-2) K_d \Lambda^{d-2}\, dl$$

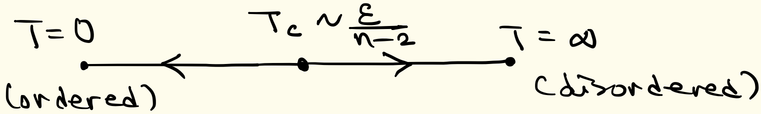
$$\Rightarrow \quad \frac{d}{dl}\left(\frac{1}{T}\right) = \frac{\varepsilon}{T} - (n-2) K_d \Lambda^{d-2}$$

$$\Rightarrow \quad \boxed{\frac{dT}{dl} = -\varepsilon T + \frac{(n-2)\,T^2}{2\pi}}$$

where we have set $d=2$ in $K_d \Lambda^{d-2}$ ($K_2 = \frac{1}{2\pi}$).

The above equation has a lot of physics. One finds a stable fixed point at $T = 0$ and an unstable fixed point at $T = \frac{2\pi\varepsilon}{(n-2)}$.

The simplest RG flow consistent with this observation is:



This suggests that

(a) In $d>2$, one always has an order-disorder transition. The RG eigenvalue for T at the unstable fixed point (i.e. at T_c) is $y_T = \nu^{-1} = \varepsilon$.

(b) In $d=2$, for $n>2$, $\frac{dT}{dl} \sim +\frac{T^2\varepsilon}{2\pi}$ i.e. an RG flow similar to that for the 1d Ising model. This implies that the system is disordered at any non-zero temperature and following the same calculation as in the 1d Ising model, the correlation length ξ at low T diverges as $\xi(T) \sim e^{\frac{2\pi}{\varepsilon T}}$.

(c) In $d = 2$, $n = 2$, $\frac{dT}{dl} = 0$ which hints at the possibility of exactly marginal operator. We will study this case separately in detail using more sophisticated analysis.